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2.(a) the following is the risk set for Q2

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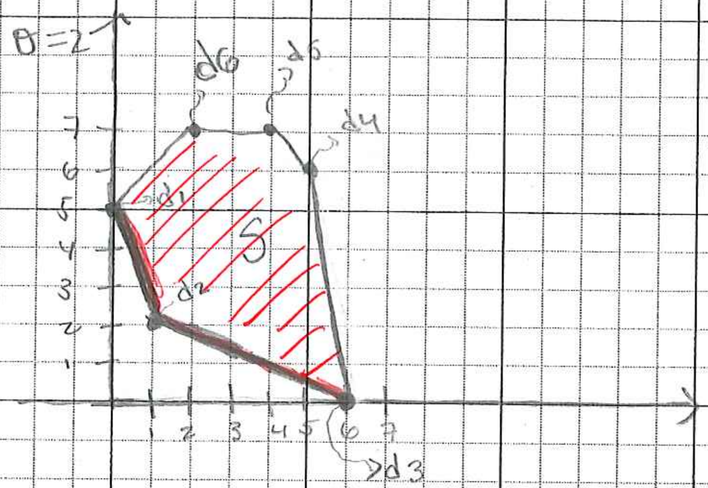

S is the risk set for D, the collection of all randomized decision rules, i.e. the convex hull of the non-randomized decision rules.

2.(b) the admissible rules are those points in the lower boundary, i.e., lines connecting d_1 with d_2 and d_2 with d_3.

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I colored this boundary red in the graph of S above.

Note that the non-randomized, admissible rules are d_1, d_2 and d_3. All others are dominated by those: $d_1 > d_6$, $d_2 > d_5$, $d_2 > d_7$, for instance.

2.(c) the minimax Rule is the intersection of the lower boundary with the line $R_1 = R_2$. Graphically:

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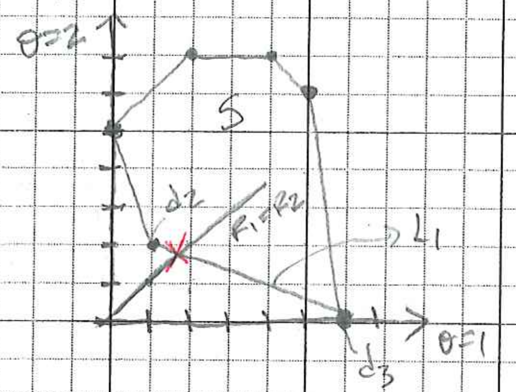

the minimax rule is marked with an ×.

Let us find this rule: First find L_1:

L_1 is the line through $(1,2)$ and $(6,0)$.

So, L_1 satisfies:

$$\begin{cases} 2 = m + b \\ 0 = 6m + b \end{cases} \Rightarrow \begin{aligned} 2 - b &= m \\ 0 &= 6(2-b) + b \\ 0 &= 12 - 6b + b \\ 5b &= 12 \Rightarrow b = 12/5 \end{aligned}$$

So, $0 = 6m + \frac{12}{5} \Rightarrow -\frac{12}{30} = m \Rightarrow m = -\frac{2}{5}$

Hence, L_1 is $y = \frac{12}{5} - \frac{2}{5}x$

the minimax rule satisfies $x = y$:

$$x = \frac{12}{5} - \frac{2}{5}x \Rightarrow \frac{7}{5}x = \frac{12}{5} \Rightarrow x = \frac{12}{7}$$

AND $y = \frac{12}{5} - \frac{2}{5}\frac{12}{7} = \frac{12}{5} - \frac{24}{35} = \frac{84-35}{35} = \frac{49}{35} = \frac{7}{5} \Rightarrow \left(\frac{12}{7}, \frac{7}{5}\right)$

So the minimax rule is the rule d^* (what λ?) is a convex combination of d_2, d_3 with weights $\lambda, (1-\lambda)$ so that $d^* = \lambda d_2 + (1-\lambda) d_3$, corresponds to the point $\left(\frac{12}{7}, \frac{7}{5}\right)$

(1) (a). Let us try to construct a UMP test for
5 $H_0: \theta \in \{1\}$ vs $H_1: \theta \in \{2,3\}$.

If we reject H_0 when $x=3$, then $P(\text{Type I error}) = P_{\theta=1}(x=3) = 0.05$.
If we reject H_0 when $x=4$, then $P(\text{Type I error}) = P_{\theta=1}(x=4) = 0.05$.
Consider the test $\phi_1(x) = \begin{cases} 1 & \text{if } x=3,4 \\ 0 & \text{if } x=1,2,5 \end{cases}$; So ϕ_1 has size $0.05+0.05=0.10$.

But this is not the only level-0.10 test, there are two others: $\phi_2(x) = \begin{cases} 1 & \text{if } x=3,5 \\ 0 & \text{if } x=1,2,4 \end{cases}$ and $\phi_3(x) = \begin{cases} 1 & \text{if } x=4,5 \\ 0 & \text{if } x=1,2,3 \end{cases}$. thus, ϕ_1, ϕ_2 & ϕ_3 are all level-0.10 tests.

The question is now: is any of these MORE powerfull than the others?
Let us compute the power of each of these:
power $\phi_1(x) = E_\theta \phi_1(x)$, where $\theta \in \{2,3\}$, so $E_{\theta=2}\phi_1(x) = P_{\theta=2}(x=3) + P_{\theta=2}(x=4) = 0.1+0.2 = 0.3$
& $E_{\theta=3}(x) = P_{\theta=3}(x=3) + P_{\theta=3}(x=4) = 0.4+0.2 = 0.6$. SAME FOR ϕ_2, ϕ_3.
Power of $\phi_2(x)$: $E_{\theta=2}\phi_2(x) = P_{\theta=2}(X=3) + P_{\theta=2}(X=5) = 0.1+0.25 = 0.35$ and
$E_{\theta=3}\phi_2(x) = P_{\theta=3}(X=3) + P_{\theta=3}(X=5) = 0.4+01 = 0.5$, AND FINALLY:
Power of $\phi_3(x)$: $E_{\theta=2}\phi_3(x) = P_{\theta=2}(X=4) + P_{\theta=2}(X=5) = 0.2+0.25 = 0.45$
$E_{\theta=3}\phi_3(x) = P_{\theta=3}(x=4) + P_{\theta=3}(X=5) = 0.2+0.1 = 0.3$. The following table summerizes the data:

Power	ϕ_1	ϕ_2	ϕ_3
$\theta=2$	0.3	0.35	0.45
$\theta=3$	0.6	0.5	0.3

So there is NO UMP, Test ϕ_3 is more powerfull for $\theta=2$ but less for $\theta=3$, while ϕ_2 is more powerfull than ϕ_1 for $\theta=2$ but not for ϕ_1, and so on.

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(1) (b) 10 $\pi(\theta=1)=0.5$, $\pi(\theta=2)=0.25$, $\pi(\theta=3)=0.25$.
By definition: $\pi(\theta|x) = \dfrac{\pi(\theta)\pi(x|\theta)}{\sum_{\theta\in\Theta} \pi(\theta')\pi(x|\theta')d\theta'}$. We are given $\pi(\theta)$. Now, if
$x=3$: denominator $= \sum_{\theta\in\{1,2,3\}} \pi(x=3|\theta)\pi(\theta) = \pi(x=3|\theta=1)\pi(\theta=1) + \pi(x=3|\theta=2)\pi(\theta=2) + \pi(x=3|\theta=3)\pi(\theta=3)$
$= 0.05\times0.5 + 0.1\times0.25 + 0.4\times0.25 = 0.15$. So, $\pi(\theta|X=3)$ is, for each $\theta\in\{1,2,3\}$
$\pi(\theta=1|x=3) = (\pi(\theta=1)\pi(x=3|\theta=1))/0.15 = (0.5\times0.05)/0.15 = \frac{1}{6}$
$\pi(\theta=2|x=3) = (\pi(\theta=2)\pi(x=3|\theta=2))/0.15 = (0.25\times0.10)/0.15 = \frac{1}{6}$
$\pi(\theta=3|x=3) = (\pi(\theta=3)\pi(x=3|\theta=3))/0.15 = (0.25\times0.4)/0.15 = \frac{4}{6}$
[Note this is a proper distribution b/c $\frac{1}{6}+\frac{1}{6}+\frac{4}{6}=1$]
Very likely, if $x=3$, that a Bayesian (or other reasonable people!) infers $\theta=3$.

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(1) 5 (c) Let us find the Bayes rule d_π. By definition, we wish to minimize:
$\sum_{\theta\in\{1,2,3\}} L(\theta, d(x))\pi(\theta)$ Equivalently, find $\frac{1}{2}R(1,d) + \frac{1}{4}R(2,d) + \frac{1}{4}R(3,d) = c$, so that c is minimum:
no, $\pi(\theta|x=3)$
$\frac{1}{2}(1-a)^2 + \frac{1}{4}(2-a)^2 + \frac{1}{4}(3-a)^2 = \frac{1}{2}[1-2a+a^2] + \frac{1}{4}[4-4a+a^2] + \frac{1}{4}[9-6a+a^2]$
$= a^2 - 2a - \frac{6}{4}a + \frac{1}{2} + 1 + \frac{9}{4} = a^2 - a(2+\frac{6}{4}) + \frac{2+4+9}{4} = a^2 - \frac{14}{4}a + \frac{15}{4} = a^2 - \frac{7}{2}a + \frac{15}{4}$, minimize this
$f(a) = a^2 - \frac{7}{2}a + \frac{15}{4}$; $f'(a_0) = 2a_0 - \frac{7}{2} = 0 \Rightarrow a_0 = \frac{7}{4}$ [This is a quadratic upwards, so this is the global min]
So the rule $d_\pi(x)$ satisfies $d_\pi(x)^2 - \frac{7}{2}d_\pi(x) + \frac{15}{4} = \frac{7}{4}$